

Structure of Polyzetas and Explicit Representation on Transcendence Bases of Shuffle and Stuffle Algebras

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Abstract

Polyzetas, indexed by words, satisfy shuffle and quasi-shuffle identities. In this respect, one can explore the multiplicative and algorithmic (locally finite) properties of their generating series. In this paper, we construct pairs of bases in duality on which polyzetas are established in order to compute local coordinates in the infinite dimensional Lie groups where their non-commutative generating series live. We also propose new algorithms leading to the ideal of polynomial relations, homogeneous in weight, among polyzetas (the graded kernel) and their explicit representation (as data structures) in terms of irreducible elements.

Keywords: Poincaré-Birkhoff-Witt basis; transcendence basis; Schützenberger's factorization; noncommutative generating series; shuffle algebra; polyzetas.

1. Introduction

This paper will provide transparent arguments and proofs for results presented at the International Symposium on Symbolic and Algebraic Computation conference, Bath, 6-9 July, 2015 [?].

For any composition of positive integers, $s = (s_1, \dots, s_r)$, the polyzetas [?] (also called multiple zeta values [?]) are defined by the following convergent series

$$\zeta(s_1, \dots, s_r) := \sum_{n_1 > \dots > n_r > 0} n_1^{-s_1} \dots n_r^{-s_r}, \quad \text{for } s_1 > 1. \quad (1)$$

The \mathbb{Q} -algebra generated by convergent polyzetas is denoted by \mathcal{Z} .

Any composition $s \in (\mathbb{N}_+)^r$ can be associated to words [? ?] of the form $x_0^{s_1-1} x_1 \dots x_0^{s_r-1} x_1$, defined on the alphabet $X = \{x_0, x_1\}$, or the form $y_{s_1} \dots y_{s_r}$,

defined on the alphabet $Y = \{y_s\}_{s \geq 1}$. The free monoids on these alphabets are respectively denoted by X^* and Y^* . In this respect, the weight of the composition s , determined by $s_1 + \dots + s_r$, is also the weight of the word $y_{s_1} \dots y_{s_r}$ or the length of the word $x_0^{s_1-1} x_1 \dots x_0^{s_r-1} x_1$.

Using concatenation, shuffle and quasi-shuffle products, in Section 2,

1. We will recall the definition of Hopf algebras $(\mathbb{Q}\langle X \rangle, \bullet, 1_{X^*}, \Delta_{\sqcup}, e)$ and $(\mathbb{Q}\langle Y \rangle, \bullet, 1_{Y^*}, \Delta_{\boxplus}, e)$.
2. Equipping X with the (total) ordering $x_0 < x_1$ and denoting by $\mathcal{Lyn}X$, the set of Lyndon words over X , the Poincaré-Birkhoff-Witt (PBW) basis $\{P_w\}_{w \in X^*}$ will be expanded over the basis $\{P_l\}_{l \in \mathcal{Lyn}X}$, of the free Lie algebra $\mathcal{Lie}_{\mathbb{Q}}\langle X \rangle$. Its dual basis $\{S_w\}_{w \in X^*}$ contains the pure transcendence basis of the algebra $(\mathbb{Q}\langle X \rangle, \sqcup, 1_{X^*})$ denoted by $\{S_l\}_{l \in \mathcal{Lyn}X}$ [?].
3. Similarly, equipping Y with the (total) ordering $y_1 > y_2 > y_3 > \dots$ and denoting by $\mathcal{Lyn}Y$ the set of Lyndon words over Y , the basis $\{\Pi_l\}_{l \in \mathcal{Lyn}Y}$, of the Lie algebra of primitive elements¹, and its associated PBW-basis $\{\Pi_w\}_{w \in Y^*}$ will be proposed. The dual basis $\{\Sigma_w\}_{w \in Y^*}$ is polynomial and contains also a pure transcendence basis of the algebra $(\mathbb{Q}\langle Y \rangle, \boxplus, 1_{Y^*})$ denoted by $\{\Sigma_l\}_{l \in \mathcal{Lyn}Y}$ [? ? ?].
4. We then establish the two following expressions of the diagonal series

$$\mathcal{D}_X := \sum_{w \in X^*} w \otimes w = \prod_{l \in \mathcal{Lyn}X}^{\rightarrow} \exp(S_l \otimes P_l), \quad (2)$$

$$\mathcal{D}_Y := \sum_{w \in Y^*} w \otimes w = \prod_{l \in \mathcal{Lyn}Y}^{\rightarrow} \exp(\Sigma_l \otimes \Pi_l). \quad (3)$$

From these, in Section 3,

1. We will consider two generating series of polyzetas² [? ? ? ?]:

$$Z_{\sqcup} := \prod_{l \in \mathcal{Lyn}X \setminus X}^{\rightarrow} \exp(\zeta(S_l)P_l) \quad \text{and} \quad Z_{\boxplus} := \prod_{l \in \mathcal{Lyn}Y \setminus \{y_1\}}^{\rightarrow} \exp(\zeta(\Sigma_l)\Pi_l). \quad (4)$$

The coefficients of Z_{\sqcup} (resp. Z_{\boxplus}) are obtained as the finite parts of the asymptotic expansions of the polylogarithms $\{\text{Li}_w\}_{w \in X^*}$ (resp. the harmonic sums $\{H_w\}_{w \in Y^*}$), at 1 (resp. at $+\infty$), in the scale of comparison

¹ P is a primitive element if $\Delta_{\boxplus}(P) = 1_{Y^*} \otimes P + P \otimes 1_{Y^*}$. This Lie algebra is isomorphic (but not equal) to the free Lie algebra.

²In (4), only *convergent* polyzetas arise then will not need any regularization.

$\{(1-z)^a \log^b((1-z)^{-1})\}_{a \in \mathbb{Z}, b \in \mathbb{N}}$ (resp. $\{N^a H_1^b(N)\}_{a \in \mathbb{Z}, b \in \mathbb{N}}$, where $H_1(N)$ is the classic harmonic sum $1 + 1/2 + \dots + 1/N$) [?].

2. We have also defined a third one, Z_γ [?], which satisfies, via the \mathfrak{u} - extended Schützenberger’s factorization on the completed quasi-shuffle Hopf algebra [? ?],

$$Z_\gamma = e^{\gamma y_1} Z_{\mathfrak{u}}. \quad (5)$$

The coefficients of Z_γ are obtained as the finite parts of the asymptotic expansions of $\{H_w\}_{w \in Y^*}$, in the scale of comparison $\{N^a \log^b(N)\}_{a \in \mathbb{Z}, b \in \mathbb{N}}$. In (5), γ denotes the Euler’s constant [?].

3. In order to identify the local coordinates of $Z_{\mathfrak{u}}$ (and $Z_{\mathfrak{u}}$), on a group of associators [? ?], we will rely on the following comparison (see [?])

$$Z_\gamma = B(y_1) \pi_Y(Z_{\mathfrak{u}}), \text{ where } B(y_1) = \exp\left(\gamma y_1 - \sum_{k \geq 2} \frac{(-1)^{k-1} \zeta(k)}{k} y_1^k\right). \quad (6)$$

Here, π_Y is a linear projection from $\mathbb{Q} \oplus \mathbb{Q}\langle\langle X \rangle\rangle_{x_1}$ to $\mathbb{Q}\langle\langle Y \rangle\rangle$, mapping $x_0^{s_1-1} x_1 \dots x_0^{s_r-1} x_1$ to $y_{s_1} \dots y_{s_r}$, and π_X denotes its inverse.

By *cancellation* [? ?], (5) and (6) yield the following identity

$$Z_{\mathfrak{u}} = B'(y_1) \pi_Y(Z_{\mathfrak{u}}), \text{ where } B'(y_1) = \exp\left(\sum_{k \geq 2} \frac{(-1)^{k-1} \zeta(k)}{k} y_1^k\right). \quad (7)$$

4. Simultaneously, algorithms will be also implemented in Maple to represent polyzetas³ in terms of irreducible polyzetas producing *algebraic* relations among the local coordinates $\{\zeta(S_l)\}_{l \in \mathcal{L}_{yn} X \setminus X}$ (and $\{\zeta(\Sigma_l)\}_{l \in \mathcal{L}_{yn} Y \setminus \{y_1\}}$) [?].

To end this section, let us point out some crucial points of our purpose :

1. Similar tables⁴ for $\{\zeta(l)\}_{l \in \mathcal{L}_{yn} X \setminus X}$ have been obtained up to weight 10 [?], 12 [?] and 16 [?]. These differ from the zig-zag relation among the moulds of formal polyzetas, due to Ecalle [?], *i.e.* the commutative generating series of *symbolic* polyzetas (Boutet de Monvel [?] and Racinet

³The Maple program runs on a computer Core(TM)i5-4210U CPU @ 1.70GHz and obtains results up to weight 12 [?].

⁴They form a Gröbner basis of the ideal of polynomial relations among the convergent polyzetas and the ranking of this basis is based mainly on the order of Lyndon words [? ? ?]. For that, this basis is also called Gröbner-Lyndon basis.

[?] have also given equivalent relations for the noncommutative generating series of symbolic polyzetas, see also [?] producing *linear* relations and which base themselves on *regularized double shuffle relation* [?] and different from identities among associators, due to Drinfel'd [?].

2. In the classical theory of finite-dimensional Lie groups, any ordered basis of Lie algebra provides a system of local coordinates in suitable neighborhood of the group unity via an ordered product of one-parameter groups corresponding to the ordered basis [?]. In this work, we get a perfect analogue of this picture for Hausdorff groups, through *Schützenberger's factorization*, this doesn't depend on regularization (see the next remark) [?]. Moreover, through the bridge equation (6) relating two elements on these groups and by identification of local coordinates, in infinite dimension, of their L.H.S. and R.H.S. (which involve only convergent polyzetas) we get again a confirmation of Zagier's conjecture, up to weight 12. This is not a consequence of regularized double-shuffle relation (see the next remarks).
3. Of course, the generating series given in (4) and (5) induce, as already shown in [?], three morphisms of (shuffle and quasi-shuffle) algebras, studied earlier in [?] and constructed in [?]

$$\zeta_{\sqcup} : (\mathbb{Q}\langle X \rangle, \sqcup, 1_{X^*}) \longrightarrow (\mathcal{Z}, \times, 1), \quad (8)$$

$$\zeta_{\sqcup} : (\mathbb{Q}\langle Y \rangle, \sqcup, 1_{Y^*}) \longrightarrow (\mathcal{Z}, \times, 1), \quad (9)$$

$$\gamma_{\bullet} : (\mathbb{Q}\langle Y \rangle, \sqcup, 1_{Y^*}) \longrightarrow (\mathcal{Z}, \times, 1), \quad (10)$$

which satisfy, for any $u = x_0^{s_1-1} x_1 \dots x_0^{s_r-1} x_1 \in x_0 X^* x_1$ and $v = \pi_Y(u)$,

$$\zeta_{\sqcup}(u) = \zeta_{\sqcup}(v) = \gamma_v = \zeta(s_1, \dots, s_r) \quad (11)$$

and the generators of length (resp. weight) one, for X^* (resp. Y^*), satisfy (see (4) and (5))

$$\zeta_{\sqcup}(x_0) = \zeta_{\sqcup}(x_1) = \zeta_{\sqcup}(y_1) = 0 \quad \text{and} \quad \gamma_{y_1} = \gamma. \quad (12)$$

Hence, $\zeta_{\sqcup}, \zeta_{\sqcup}$ and γ_{\bullet} are characters of (shuffle and quasi-shuffle) Hopf algebras, and their graphs, written as series, respectively read [?]

$$\sum_{w \in X^*} \zeta_{\sqcup}(w)w = Z_{\sqcup}, \quad \sum_{w \in Y^*} \zeta_{\sqcup}(w)w = Z_{\sqcup}, \quad \sum_{w \in Y^*} \gamma_w w = Z_{\gamma} \quad (13)$$

and⁵ $Z_{\sqcup} = (\zeta_{\sqcup} \otimes \text{Id}_{X^*})\mathcal{D}_X, Z_{\sqcup} = (\zeta_{\sqcup} \otimes \text{Id}_{Y^*})\mathcal{D}_Y, Z_{\gamma} = (\gamma_{\bullet} \otimes \text{Id}_{Y^*})\mathcal{D}_Y.$

⁵They are group-like : $\Delta_{\sqcup}(Z_{\sqcup}) = Z_{\sqcup} \otimes Z_{\sqcup}, \Delta_{\sqcup}(Z_{\sqcup}) = Z_{\sqcup} \otimes Z_{\sqcup}, \Delta_{\gamma}(Z_{\gamma}) = Z_{\gamma} \otimes Z_{\gamma}.$

4. By (4), for any $u, v \in \mathcal{Lyn} X \setminus X$ and $u' = \pi_Y(u), v' = \pi_Y(v)$, one has

$$\zeta_{\sqcup}(u)\zeta_{\sqcup}(v) = \zeta_{\sqcup}(u \sqcup v) \quad \text{and} \quad \zeta_{\boxplus}(u')\zeta_{\boxplus}(v') = \zeta_{\boxplus}(u' \boxplus v'). \quad (14)$$

By (7), for any $l \in \mathcal{Lyn} X \setminus X$ and $l' = \pi_Y(l)$, one has, on the other hand

- i) $\zeta_{\sqcup}(x_1 \sqcup l - x_1 l) = -\zeta_{\sqcup}(x_1 l) = -\langle Z_{\sqcup} \mid x_1 l \rangle,$
- ii) $\zeta_{\boxplus}(y_1 \boxplus l' - y_1 l') = -\zeta_{\boxplus}(y_1 l') = -\langle Z_{\boxplus} \mid y_1 l' \rangle,$
- iii) $\langle B'(y_1) \mid y_1 \rangle = 0.$

This means that since (7) is equivalent to (6), for the quasi-shuffle product, the regularization to γ is equivalent to the regularization to 0 [? ?] and this yields immediately the family of regularized double shuffle relations considered in [? ? ? ? ?] (see also [? ? ? ? ?]).

Our method is then different from [? ? ?] in which their authors suggest the *simultaneous* regularization of the divergent polyzeta $\zeta(1)$ to the indeterminate T , *i.e.* $\zeta_{\sqcup}(x_0) = \zeta_{\sqcup}(x_1) = \zeta_{\boxplus}(y_1) = T$ (to compare with (12)). Since T is transcendent over \mathbb{Q} then it can be suitable to be specialized to 0, as effectively done in [? ?] and, by this way, relations among polyzetas are *formally* obtained depending mainly on numerical values⁶ of T .

2. Background

2.1. Generalities

Let $Y = \{y_s\}_{s \geq 1}$ be an infinite alphabet with the total order $y_1 > y_2 > \dots$. Y^* denotes the free monoid on Y which admits the empty word, denoted by 1_{Y^*} , as neutral element.

Let us define the commutative product on⁷ $\mathbb{Q}Y$, denoted by μ (see [? ?]),

$$\forall y_s, y_t \in Y, \quad \mu(y_s, y_t) = y_{s+t}, \quad (15)$$

or its dual coproduct, Δ_μ , defined by

$$\forall y_s \in Y, \quad \Delta_\mu y_s = \sum_{i=1}^{s-1} y_i \otimes y_{s-i} \quad (16)$$

satisfying,

$$\forall x, y, z \in Y, \quad \langle \Delta_\mu x \mid y \otimes z \rangle = \langle x \mid \mu(y, z) \rangle. \quad (17)$$

Let $\mathbb{Q}\langle Y \rangle$ denote the space of polynomials on the alphabet Y equipped by

⁶Since the \mathbb{Q} -algebra of polyzetas is not a $\mathbb{Q}[T]$ -algebra, how then can we determine these values ?

⁷ $\mathbb{Q}Y$ denotes the \mathbb{Q} -vector space generated by the alphabet Y , as a basis.

1. The concatenation \bullet (or by its associated coproduct, Δ_\bullet).
2. The *shuffle* product, *i.e.* the commutative product defined by [?], for any $y_s, y_t \in Y$ and $u, v, w \in Y^*$

$$\begin{aligned} w \sqcup 1_{Y^*} &= 1_{Y^*} \sqcup w = w, \\ y_s u \sqcup y_t v &= y_s(u \sqcup y_t v) + y_t(y_s u \sqcup v) \end{aligned} \quad (18)$$

or by its associated coproduct, Δ_\sqcup , defined, on the letters by,

$$\forall y_s \in Y, \quad \Delta_\sqcup y_s = y_s \otimes 1_{Y^*} + 1_{Y^*} \otimes y_s \quad (19)$$

and extended so as to make it a homomorphism for the concatenation product. It satisfies

$$\forall u, v, w \in Y^*, \quad \langle \Delta_\sqcup w \mid u \otimes v \rangle = \langle w \mid u \sqcup v \rangle. \quad (20)$$

3. The *quasi-shuffle* product, *i.e.* the commutative product defined by [?], for any $y_s, y_t \in Y$ and $u, v, w \in Y^*$,

$$\begin{aligned} w \boxplus 1_{Y^*} &= 1_{Y^*} \boxplus w = w, \\ y_s u \boxplus y_t v &= y_s(u \boxplus y_t v) + y_t(y_s u \boxplus v) + \mu(y_s, y_t)(u \boxplus v) \end{aligned} \quad (21)$$

or by its associated coproduct, Δ_\boxplus , defined, on the letters by,

$$\forall y_s \in Y, \quad \Delta_\boxplus y_s = \Delta_\sqcup y_s + \Delta_\mu y_s \quad (22)$$

and extended so as to make it a homomorphism for the concatenation product. It satisfies

$$\forall u, v, w \in Y^*, \quad \langle \Delta_\boxplus w \mid u \otimes v \rangle = \langle w \mid u \boxplus v \rangle. \quad (23)$$

Note that Δ_\sqcup and Δ_\boxplus are morphisms from $\mathbb{Q}\langle Y \rangle$ for the concatenation but Δ_μ is not (for example $\Delta_\mu(y_1^2) = y_1 \otimes y_1$, whereas $\Delta_\mu(y_1)^2 = 0$).

Hence, with the counit ϵ defined by $\epsilon(P) = \langle P \mid 1_{Y^*} \rangle$ (for any $P \in \mathbb{Q}\langle Y \rangle$). We get two pairs of mutually dual bialgebras

$$\mathcal{H}_\sqcup = (\mathbb{Q}\langle Y \rangle, \bullet, 1_{Y^*}, \Delta_\sqcup, \epsilon), \quad \mathcal{H}_\sqcup^\vee = (\mathbb{Q}\langle Y \rangle, \sqcup, 1_{Y^*}, \Delta_\bullet, \epsilon), \quad (24)$$

$$\mathcal{H}_\boxplus = (\mathbb{Q}\langle Y \rangle, \bullet, 1_{Y^*}, \Delta_\boxplus, \epsilon), \quad \mathcal{H}_\boxplus^\vee = (\mathbb{Q}\langle Y \rangle, \boxplus, 1_{Y^*}, \Delta_\bullet, \epsilon). \quad (25)$$

Let us then consider the following diagonal series⁸

$$\mathcal{D}_{\sqcup} = \sum_{w \in Y^*} w \otimes w \quad \text{and} \quad \mathcal{D}_{\sqcup\sqcup} = \sum_{w \in Y^*} w \otimes w. \quad (26)$$

Here, for the algebras where live in \mathcal{D}_{\sqcup} and $\mathcal{D}_{\sqcup\sqcup}$, the operation on the right factor of the tensor product is the concatenation, and the operation on the left factor is the shuffle and the quasi-shuffle, respectively.

By the Cartier-Quillen-Milnor and Moore (CQMM) theorem [? ?], the connected \mathbb{N} -graded, co-commutative Hopf algebra \mathcal{H}_{\sqcup} is isomorphic to the enveloping algebra of the Lie algebra of its primitive elements which is $\mathcal{Lie}_{\mathbb{Q}}\langle Y \rangle$:

$$\mathcal{H}_{\sqcup} \cong \mathcal{U}(\mathcal{Lie}_{\mathbb{Q}}\langle Y \rangle) \quad \text{and} \quad \mathcal{H}_{\sqcup}^{\vee} \cong \mathcal{U}(\mathcal{Lie}_{\mathbb{Q}}\langle Y \rangle)^{\vee}. \quad (27)$$

Hence, denoting by (l_1, l_2) the standard factorization⁹ of $l \in \mathcal{Lyn}Y \setminus Y$, let us consider

1. The PBW basis $\{P_w\}_{w \in Y^*}$ constructed recursively as follows [?]

$$\begin{cases} P_{y_s} &= y_s & \text{for } y_s \in Y, \\ P_l &= [P_{l_1}, P_{l_2}] & \text{for } l \in \mathcal{Lyn}Y \setminus Y, \text{ st}(l) = (l_1, l_2), \\ P_w &= P_{l_1}^{i_1} \dots P_{l_k}^{i_k} & \text{for } w = l_1^{i_1} \dots l_k^{i_k}, \text{ with} \\ & & l_1, \dots, l_k \in \mathcal{Lyn}Y, l_1 > \dots > l_k. \end{cases} \quad (28)$$

Example 1. *i) Considering on the alphabet Y :*

$$\begin{aligned} P_{y_1} &= y_1, & P_{y_2} &= y_2, \\ P_{y_2 y_1} &= y_2 y_1 - y_1 y_2, \\ P_{y_3 y_1 y_2} &= y_3 y_1 y_2 - y_2 y_3 y_1 + y_2 y_1 y_3 - y_1 y_3 y_2. \end{aligned}$$

ii) Considering on the alphabet $X = \{x_0, x_1\}$, $x_0 < x_1$:

$$\begin{aligned} P_{x_1} &= x_1, & P_{x_0 x_1} &= x_0 x_1 - x_1 x_0, \\ P_{x_0 x_1^2} &= x_0 y_1^2 - 2x_1 x_0 x_1 + y_1^2 x_0, \\ P_{x_0^2 x_1^2 x_0 x_1} &= x_0^2 x_1^2 x_0 x_1 - x_0^2 x_1^3 x_0 + 2x_0 x_1 x_0 x_1^2 x_0 + 2x_1 x_0 x_1 x_0 x_0 x_1 \\ &\quad - x_1^2 x_0^3 x_1 + x_1^2 x_0^2 x_1 x_0 - x_0 x_1 x_0^2 x_1^2 - 2x_0 x_1^2 x_0 x_1 x_0 + x_0 x_1^3 x_0^2 \\ &\quad + x_1 x_0^3 x_1^2 - 2x_1 x_0^2 x_1 x_0 x_1 - x_1 x_0 x_1^2 x_0^2. \end{aligned}$$

⁸Of course, we have (set theoretically) $\mathcal{D}_{\sqcup} = \mathcal{D}_{\sqcup\sqcup}$, but their structural treatments will be different.

⁹A pair of Lyndon words (l_1, l_2) is called the standard factorization of l if $l = l_1 l_2$ and l_2 is the smallest nontrivial proper right factor of l (for the lexicographic order) or, equivalently its longest such.

2. and, by duality¹⁰, the basis $\{S_w\}_{w \in Y^*}$ of $(\mathbb{Q}\langle Y \rangle, \sqcup)$, i.e.

$$\forall u, v \in Y^*, \langle P_u \mid S_v \rangle = \delta_{u,v}. \quad (29)$$

This linear basis can be computed recursively as follows [?]

$$\begin{cases} S_{y_s} &= y_s, & \text{for } y_s \in Y, \\ S_l &= y_s S_u, & \text{for } l = y_s u \in \mathcal{L}yn Y, \\ S_w &= \frac{S_{l_1}^{i_1} \sqcup \dots \sqcup S_{l_k}^{i_k}}{i_1! \dots i_k!} & \text{for } w = l_1^{i_1} \dots l_k^{i_k}, \text{ with} \\ & & l_1, \dots, l_k \in \mathcal{L}yn Y, l_1 > \dots > l_k. \end{cases} \quad (30)$$

Example 2. *i) Considering on the alphabet Y :*

$$\begin{aligned} S_{y_1} &= y_1, \\ S_{y_2} &= y_2, \\ S_{y_2 y_1} &= y_2 y_1, \\ S_{y_3 y_1 y_2} &= y_3 y_2 y_1 + y_3 y_1 y_2. \end{aligned}$$

ii) Considering on the alphabet X :

$$\begin{aligned} S_{x_1} &= x_1, \\ S_{x_0 x_1} &= x_0 x_1, \\ S_{x_0 x_1^2} &= x_0 x_1^2, \\ S_{x_0^2 x_1^2 x_0 x_1} &= x_0^2 x_1^2 x_0 x_1 + 3x_0^2 x_1 x_0 x_1^2 + 6x_0^3 x_1^3. \end{aligned}$$

Similarly, by CQMM theorem, the connected \mathbb{N} -graded, co-commutative Hopf algebra \mathcal{H}_{\sqcup} is isomorphic to the enveloping algebra of its primitive elements:

$$\text{Prim}(\mathcal{H}_{\sqcup}) = \text{Im}(\pi_1) = \text{span}_{\mathbb{Q}}\{\pi_1(w) \mid w \in Y^*\}, \quad (31)$$

where, for any $w \in Y^*$, $\pi_1(w)$ is obtained as follows [? ?]

$$\pi_1(w) = w + \sum_{k=2} \frac{(-1)^{k-1}}{k} \sum_{u_1, \dots, u_k \in Y^+} \langle w \mid u_1 \sqcup \dots \sqcup u_k \rangle u_1 \dots u_k. \quad (32)$$

¹⁰The dual family, i.e. the set of coordinates forming a basis in the algebraic dual which is here the space of noncommutative series, but as the enveloping algebra under consideration is graded in finite dimensions (by the multidegree), these series are in fact multi-homogeneous polynomials.

Note that (32) is equivalent to the following identity

$$w = \sum_{k \geq 0} \frac{1}{k!} \sum_{u_1, \dots, u_k \in Y^*} \langle w \mid u_1 \sqcup \dots \sqcup u_k \rangle \pi_1(u_1) \dots \pi_1(u_k). \quad (33)$$

In particular, for any $y_s \in Y$, the primitive polynomial $\pi_1(y_s)$ is given by

$$\pi_1(y_s) = y_s + \sum_{i=2}^s \frac{(-1)^{i-1}}{i} \sum_{j_1, \dots, j_i \geq 1, j_1 + \dots + j_i = s} y_{j_1} \dots y_{j_i}. \quad (34)$$

Example 3. $\pi_1(y_1) = y_1$, $\pi_1(y_2) = y_2 - \frac{1}{2}y_1^2$, $\pi_1(y_3) = y_3 - \frac{1}{2}(y_1y_2 + y_2y_1) + \frac{1}{3}y_1^3$.

As previously, the expressions (34) are equivalent to

$$y_s = \sum_{i \geq 1} \frac{1}{i!} \sum_{s_1 + \dots + s_i = s} \pi_1(y_{s_1}) \dots \pi_1(y_{s_i}), \quad y_s \in Y. \quad (35)$$

Example 4.

$$\begin{aligned} y_1 &= \pi_1(y_1), \\ y_2 &= \pi_1(y_2) + \frac{1}{2!}\pi_1(y_1)^2, \\ y_3 &= \pi_1(y_3) + \frac{1}{2!}(\pi_1(y_1)\pi_1(y_2) + \pi_1(y_2)\pi_1(y_1)) + \frac{1}{3!}\pi_1(y_1)^3. \end{aligned}$$

Now let us consider the (endo-)morphism of algebras $\phi : (\mathbb{Q}\langle Y \rangle, \bullet, 1) \rightarrow (\mathbb{Q}\langle Y \rangle, \bullet, 1)$ satisfying $\phi(y_k) = \pi_1(y_k)$; it can be shown that ϕ is an automorphism of $\mathbb{Q}\langle Y \rangle$. Then we have [?],

- i) ϕ realizes an isomorphism from the bialgebra $(\mathbb{Q}\langle Y \rangle, \bullet, \Delta_{\sqcup}, e)$ to the bialgebra $(\mathbb{Q}\langle Y \rangle, \bullet, \Delta_{\sqcup}, e)$.
- ii) In particular, we have the following commutative diagram

$$\begin{array}{ccc} \mathbb{Q}\langle Y \rangle & \xrightarrow{\Delta_{\sqcup}} & \mathbb{Q}\langle Y \rangle \otimes \mathbb{Q}\langle Y \rangle \\ \phi \downarrow & & \downarrow \phi \otimes \phi \\ \mathbb{Q}\langle Y \rangle & \xrightarrow{\Delta_{\sqcup}} & \mathbb{Q}\langle Y \rangle \otimes \mathbb{Q}\langle Y \rangle. \end{array}$$

- iii) $\mathcal{H}_{\sqcup} \cong \mathcal{U}(\text{Prim}(\mathcal{H}_{\sqcup}))$ and $\mathcal{H}_{\sqcup}^{\vee} \cong \mathcal{U}(\text{Prim}(\mathcal{H}_{\sqcup}))^{\vee}$.
- iv) The dual bases $\{\Pi_w\}_{w \in Y^*}$ and $\{\Sigma_w\}_{w \in Y^*}$ of respectively $\mathcal{U}(\text{Prim}(\mathcal{H}_{\sqcup}))$ and $\mathcal{U}(\text{Prim}(\mathcal{H}_{\sqcup}))^{\vee}$ can be obtained as images, respectively by ϕ and $\check{\phi}^{-1}$, of respectively $\{P_w\}_{w \in Y^*}$ and $\{S_w\}_{w \in Y^*}$.

More precisely,

1. The PBW basis $\{\Pi_w\}_{w \in Y^*}$ for $\mathcal{U}(\text{Prim}(\mathcal{H}_{\boxplus}))$ can be constructed recursively as follows [? ? ?]

$$\begin{cases} \Pi_{y_s} &= \pi_1(y_s) & \text{for } y_s \in Y, \\ \Pi_l &= [\Pi_{l_1}, \Pi_{l_2}] & \text{for } l \in \mathcal{Lyn}Y \setminus Y, \text{ } st(l) = (l_1, l_2), \\ \Pi_w &= \Pi_{l_1}^{i_1} \dots \Pi_{l_k}^{i_k} & \text{for } w = l_1^{i_1} \dots l_k^{i_k}, \text{ with} \\ & & l_1, \dots, l_k \in \mathcal{Lyn}Y, \text{ } l_1 > \dots > l_k. \end{cases} \quad (36)$$

Example 5.

$$\begin{aligned} \Pi_{y_1} &= y_1, \\ \Pi_{y_2} &= y_2 - \frac{1}{2}y_1^2, \\ \Pi_{y_2y_1} &= y_2y_1 - y_1y_2, \\ \Pi_{y_3y_1y_2} &= y_3y_1y_2 - \frac{1}{2}y_3y_1^3 - y_2y_1^2y_2 + \frac{1}{4}y_2y_1^4 - y_1y_3y_2 + \frac{1}{2}y_1y_3y_1^2 + \frac{1}{2}y_1^2y_2^2 \\ &\quad - \frac{1}{2}y_1^2y_2y_1^2 - y_2y_3y_1 + \frac{1}{2}y_2^2y_1^2 + y_2y_1y_3 + \frac{1}{2}y_1^2y_3y_1 - \frac{1}{2}y_1^3y_3 + \frac{1}{4}y_1^4y_2. \end{aligned}$$

2. and, by duality, the basis $\{\Sigma_w\}_{w \in Y^*}$ of $(\mathbb{Q}\langle Y \rangle, \boxplus)$, i.e.

$$\forall u, v \in Y^*, \quad \langle \Pi_u \mid \Sigma_v \rangle = \delta_{u,v}. \quad (37)$$

This linear basis can be computed recursively as follows [? ? ?]

$$\begin{cases} \Sigma_{y_s} = y_s, & \text{for } y_s \in Y, \\ \Sigma_l = \sum_{(\star)} \frac{1}{i!} y_{s_{k_1} + \dots + s_{k_i}} \Sigma_{l_1 \dots l_n}, & \text{for } l = y_{s_1} \dots y_{s_k} \in \mathcal{Lyn}Y, \\ \Sigma_w = \frac{\Sigma_{l_1}^{\boxplus i_1} \boxplus \dots \boxplus \Sigma_{l_k}^{\boxplus i_k}}{i_1! \dots i_k!}, & \text{for } w = l_1^{i_1} \dots l_k^{i_k}, \text{ with} \\ & l_1, \dots, l_k \in \mathcal{Lyn}Y, \text{ } l_1 > \dots > l_k. \end{cases} \quad (38)$$

In (\star) , the sum is taken over all $\{k_1, \dots, k_i\} \subset \{1, \dots, k\}$ and all $l_1 \geq \dots \geq l_n$ such that $(y_{s_1}, \dots, y_{s_k}) \stackrel{*}{\leftarrow} (y_{s_{k_1}}, \dots, y_{s_{k_i}}, l_1, \dots, l_n)$, where $\stackrel{*}{\leftarrow}$ denotes the transitive closure of the relation on standard sequences, denoted by \leftarrow [?]. Using Example 2.ii), we have in general, for any $l \in \mathcal{Lyn}Y$, $\pi_X(\Sigma_l) \neq S_{\pi_X l}$ (resp. $\mathcal{Lyn}X \setminus \{x_0\}$, $\pi_Y(S_l) \neq \Sigma_{\pi_Y l}$) [? ?]:

Example 6.

$l \in \mathcal{Lyn}Y$	Σ_l	$\pi_X(l) \in \mathcal{Lyn}X$	$\pi_Y S_{\pi_X(l)}$
y_1	y_1	x_1	y_1
y_2	y_2	x_0x_1	y_2
y_2y_1	$y_2y_1 + \frac{1}{2}y_3$	$x_0x_1^2$	y_2y_1
$y_3y_1y_2$	$y_3y_2y_1 + y_3y_1y_2 + y_3^2$ $+ \frac{1}{2}y_4y_2 + \frac{1}{2}y_5y_1 + \frac{1}{3}y_6$	$x_0^2x_1^2x_0x_1$	$y_3y_1y_2 + 3y_3y_2y_1$ $+ 6y_4y_1^2$

2.2. Local coordinates

Following Wei-Norman's theorem [?], we know that, for a given (finite dimensional) \mathbf{k} -Lie group¹¹ G , its Lie algebra \mathfrak{g} , and a basis $B = (b_i)_{1 \leq i \leq n}$ of \mathfrak{g} , there exists a neighbourhood W of 1_G (in G) and n *local coordinate* \mathbf{k} -valued analytic functions

$$W \rightarrow \mathbf{k}, \quad (t_i)_{1 \leq i \leq n}$$

such that, for all $g \in W$,

$$g = \prod_{1 \leq i \leq n}^{\rightarrow} e^{t_i(g)b_i} = e^{t_1(g)b_1} \dots e^{t_n(g)b_n}.$$

The proof relies on the fact that, $(t_1, \dots, t_n) \rightarrow e^{t_1(g)b_1} \dots e^{t_n(g)b_n}$ is a local diffeomorphism from \mathbf{k}^n to G at a neighbourhood of 0.

Example 7 (Wei-Norman in finite dimensions). *Let $M \in Gl_+(2, \mathbb{R})$ ($Gl_+(2, \mathbb{R})$ denote the connected component of 1 in the Lie group¹² $Gl(2, \mathbb{R})$)*

$$M = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

In order to perform the decomposition, we will “go back to identity” by computing $MTDU = I$, where I stands for the identity matrix, T is upper unitriangular, D diagonal strictly positive and U unitary, then $M = U^{-1}D^{-1}T^{-1}$ will be the Iwasawa [?] decomposition of M . The decomposition algorithm goes in three steps as follows (step 4 is a summary)

1. **(Orthogonalization)** *We perform block-computation on the columns of M to obtain an orthogonal matrix*

$$M \longrightarrow \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 & t_1 \\ 0 & 1 \end{pmatrix} = MT = \begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} \\ a_{21}^{(1)} & a_{22}^{(1)} \end{pmatrix} = (C_1^{(1)} \quad C_2^{(1)}) = M_1.$$

the both of columns are orthogonal if $t_1 = -\frac{a_{11}a_{12}+a_{21}a_{22}}{a_{11}^2+a_{21}^2}$.

¹¹Real (with $\mathbf{k} = \mathbb{R}$) or complex (with $\mathbf{k} = \mathbb{C}$).

¹²It is the group of matrices with positive determinant.

2. **(Normalization)** We normalize M_1 ,

$$\begin{aligned} M_2 &= \begin{pmatrix} C_1^{(1)} & C_2^{(1)} \end{pmatrix} \begin{pmatrix} \frac{1}{\|C_1^{(1)}\|} & 0 \\ 0 & \frac{1}{\|C_2^{(1)}\|} \end{pmatrix} = M_1 D \\ &= M_1 e^{-\log(\|C_1^{(1)}\|)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} e^{-\log(\|C_2^{(1)}\|)} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

3. **(Unitarization)** As the columns of M_2 form an orthogonal basis and as $\det(M_2) > 0$, one can write

$$M_2 = \begin{pmatrix} a_{11}^{(2)} & a_{12}^{(2)} \\ a_{21}^{(2)} & a_{22}^{(2)} \end{pmatrix} = \begin{pmatrix} \cos(t_2) & -\sin(t_2) \\ \sin(t_2) & \cos(t_2) \end{pmatrix} = e^{t_2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and as M_2 is in a neighbourhood of I_2 , one has $t_2 = \arctan(\frac{a_{21}}{a_{11}})$.

4. **(Summary)**

$$MTD = M_2 = e^{\arctan(\frac{a_{21}}{a_{11}})} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

hence

$$\begin{aligned} M &= e^{\arctan(\frac{a_{21}}{a_{11}})} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} D^{-1} T^{-1} \\ &= e^{\arctan(\frac{a_{21}}{a_{11}})} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} e^{\log(\|C_1\|)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} e^{\log(\|C_2^{(1)}\|)} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} e^{\frac{\langle C_1 | C_2 \rangle}{\|C_1\|^2}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

One then gets a Wei-Norman decomposition of M with respect to the basis of the Lie algebra $\mathfrak{gl}(2, \mathbb{R})$: $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Now, in infinite dimensions, *i.e.* here within the algebra of double series (whose support is a subset of $Y^* \otimes Y^*$) endowed with the law $\sqcup \hat{\otimes} \bullet$, we have Schützenberger's factorization(s) [? ?] as a perfect analogue of Wei-Norman's theorem for the group of group-like series. For \mathcal{D}_{\sqcup}

$$\mathcal{D}_{\sqcup} = \prod_{l \in \mathcal{L}_{yn} Y}^{\searrow} \exp(S_l \otimes P_l) \in \mathcal{H}_{\sqcup}^{\vee} \hat{\otimes} \mathcal{H}_{\sqcup};$$

or with the law $\sqcup \hat{\otimes} \bullet$, we also have the extension of Schützenberger's factorization for \mathcal{D}_{\sqcup} which is then [? ? ?]

$$\mathcal{D}_{\sqcup} = \prod_{l \in \mathcal{L}_{yn} Y}^{\searrow} \exp(\Sigma_l \otimes \Pi_l) \in \mathcal{H}_{\sqcup}^{\vee} \hat{\otimes} \mathcal{H}_{\sqcup}.$$

These can be used to provide a system of local coordinates on the Hausdorff group (*i.e.* group of group-like elements¹³). Applying these factorizations to the multiple zeta functions $\zeta_{\sqcup}, \zeta_{\sqcup\sqcup}$, or to Z_{\sqcup} and $Z_{\sqcup\sqcup}$ (which are all group-like), we have the representations

$$Z_{\sqcup} = \prod_{l \in \mathcal{L}ynX \setminus X}^{\rightarrow} e^{\zeta(S_l)P_l} \quad \text{and} \quad Z_{\sqcup\sqcup} = \prod_{l \in \mathcal{L}ynY \setminus \{y_1\}}^{\rightarrow} e^{\zeta(\Sigma_l)\Pi_l}.$$

It means that all relations among polyzetas which can be seen here will be taken from relations among their local coordinates. Our method is to use identity (7) to reduce relations between the two systems of local coordinates $\{\zeta(S_l)\}_{l \in \mathcal{L}ynX}$ and $\{\zeta(\Sigma_l)\}_{l \in \mathcal{L}ynY}$.

3. Structure of polyzetas

3.1. Representations of polynomials on bases

The aim of this subsection is to provide a method to represent any polynomial of $\mathbb{Q}\langle Y \rangle$ in terms of each basis $\{P_w\}_{w \in Y^*}$, $\{S_w\}_{w \in Y^*}$, $\{\Pi_w\}_{w \in Y^*}$ or $\{\Sigma_w\}_{w \in Y^*}$.

Recall that the bases $\{P_w\}_{w \in Y^*}$ and $\{\Pi_w\}_{w \in Y^*}$ are homogeneous and upper triangular, the bases $\{S_w\}_{w \in Y^*}$ and $\{\Sigma_w\}_{w \in Y^*}$ are homogeneous and lower triangular¹⁴. Without loss of generality we can assume that $P \in \mathbb{Q}\langle Y \rangle$ is a homogeneous polynomial of weight n , we now represent P in terms of the basis $\{\Sigma_w\}_{w \in Y^*}$ by the following algorithm.

Algorithm 1

INPUT: A homogeneous polynomial P of weight n .

OUTPUT: The representation of P in terms of the basis $\{\Sigma_w\}_{w \in Y^*}$.

Step 1. We choose the leading term¹⁵ of P , assumed $\lambda_1 w_1$. Expressing the word w_1 as follows

$$w_1 = \Sigma_{w_1} + \sum_{v < w_1, (v)=n} \alpha_v v. \quad (39)$$

The polynomial P can now be rewritten in the form

$$P = \lambda_{w_1} \Sigma_{w_1} + \sum_{v < w_1, (v)=n} \beta_v v. \quad (40)$$

¹³In fact, these series are respectively characters for \sqcup or $\sqcup\sqcup$.

¹⁴w.r.t the words and the lexicographic ordering, for example, $\Sigma_w = w + \sum_{v < w, (v)=(w)} \alpha_v v$.

¹⁵This term includes the greatest word in the support of P and its coefficient.

Step 2. We repeat **Step 1** with P now understood as the polynomial $\sum_{v < w_1, (v)=n} \beta_v v$, and so on until the last monomial which admits the smallest word of weight n , y_n , and we really have $y_n = \Sigma_{y_n}$. At last, by re-expressing the coefficients, we will obtain the representation of the original in form that

$$P = \sum_{v \leq w_1, (v)=n} \lambda_v \Sigma_v. \quad (41)$$

Example 8. $P := 2y_1y_2 - 1/2y_3$.

Step 1. Since $\Sigma_{y_1y_2} = y_1y_2 + y_2y_1 + y_3$, we replace y_1y_2 with $\Sigma_{y_1y_2} - y_2y_1 - y_3$ in P

$$P = 2\Sigma_{y_1y_2} - 2y_2y_1 - 5/2y_3.$$

Step 2. Since $\Sigma_{y_2y_1} = y_2y_1 + 1/2y_3$, we replace y_2y_1 with $\Sigma_{y_2y_1} - 1/2y_3$ in P

$$P = 2\Sigma_{y_1y_2} - 2\Sigma_{y_2y_1} - 3/2y_3.$$

Since $y_3 = \Sigma_{y_3}$, we thus get $P = 2\Sigma_{y_1y_2} - 2\Sigma_{y_2y_1} - 3/2\Sigma_{y_3}$.

Corollary 1. For any $w \in Y^*$, we can represent¹⁶

$$\begin{aligned} w &= P_w + \sum_{u > w, |u|=|w|} \alpha_u^1 P_u = S_w + \sum_{u < w, |u|=|w|} \alpha_u^2 S_u, \\ w &= \Pi_w + \sum_{v > w, (v)=(w)} \beta_v^1 \Pi_v = \Sigma_w + \sum_{v < w, (v)=(w)} \beta_v^2 \Sigma_v. \end{aligned}$$

3.2. Identifying the local coordinates

We now use the alphabet $X = \{x_0, x_1\}$ ordered by $x_0 < x_1$. Returning to formula (7), with the bases $\{P_w\}_{w \in X^*}$ and $\{S_w\}_{w \in X^*}$ defined as (28) and (30), we will find relations among polyzetas by identifying on the bases as local coordinates. First, we expand B' , given in (7), in form of generating series of y_1 .

Lemma 1. We have

$$B'(y_1) = 1 + \sum_{m \geq 2} B^{(m)} y_1^m, \quad \text{with} \quad B^{(m)} = \sum_{i=1}^{\lfloor m/2 \rfloor} \sum_{\substack{k_1, \dots, k_i \geq 2 \\ k_1 + \dots + k_i = m}} (-1)^{m-i} \frac{\zeta(k_1) \dots \zeta(k_i)}{k_1 \dots k_i},$$

where $\lfloor m/2 \rfloor$ is the largest integer not greater than $m/2$.

¹⁶ $|w|$ and (w) respectively denote the length and the weight of the word w .

Proof. Expanding the exponential, one has successively

$$\begin{aligned}
B'(y_1) &= \sum_{n \geq 0} \frac{1}{n!} \left(\sum_{k \geq 2} \frac{(-1)^{k-1} \zeta(k)}{k} y_1^k \right)^n \\
&= \sum_{n \geq 0} \frac{1}{n!} \sum_{k_1, \dots, k_n \geq 2} \frac{(-1)^{k_1 + \dots + k_n - n} \zeta(k_1) \dots \zeta(k_n)}{k_1 \dots k_n} y_1^{k_1 + \dots + k_n} \\
&= 1 + \sum_{m \geq 2} \left(\sum_{n=1}^{\lfloor m/2 \rfloor} \frac{1}{n!} \sum_{\substack{k_1, \dots, k_n \geq 2 \\ k_1 + \dots + k_n = m}} \frac{(-1)^{m-n} \zeta(k_1) \dots \zeta(k_n)}{k_1 \dots k_n} \right) y_1^m \\
&= 1 + \sum_{m \geq 2} B^{(m)} y_1^m.
\end{aligned}$$

□

Example 9.

$$\begin{aligned}
B^{(2)} &= -\frac{\zeta(2)}{2}, \\
B^{(3)} &= \frac{\zeta(3)}{3}, \\
B^{(4)} &= -\frac{\zeta(4)}{4} + \frac{\zeta(2)^2}{2^2}, \\
B^{(5)} &= \frac{\zeta(5)}{5} - 2 \frac{\zeta(2)}{2} \frac{\zeta(3)}{3}.
\end{aligned}$$

3.2.1. *Identifying with respect to the basis* $\{\Pi_w\}_{w \in Y^*}$

Using the duality of the bases, we rewrite (7) as follows

$$\sum_{v \in Y^*} \zeta_{\sqcup}(\Sigma_v) \Pi_v = B'(y_1) \sum_{v \in Y^*} \zeta_{\sqcup}(\pi_X(\Sigma_v)) \Pi_v. \quad (42)$$

Moreover, we see that $B'(y_1)$ is a series of a single letter (like a single variable), y_1 , and

$$y_1^k \Pi_v = \Pi_{y_1^k} \Pi_v = \Pi_{y_1^k v}, \quad \forall k \geq 1, v \in Y^*.$$

We can then identify the coefficients in (42) and obtain:

Proposition 1. *i) For any $v \in Y^* \setminus y_1 Y^*$, one has¹⁷ $\zeta(\Sigma_v) = \zeta(\pi_X \Sigma_v)$.*

¹⁷As $x_0 X^* x_1$ and $Y^* \setminus y_1 Y^*$ are disjointed, the unique notation $\zeta(P)$ is used here to replace $\zeta_{\sqcup}(P)$ or $\zeta_{\sqcup}(P)$ if the polynomial P only contains convergent words.

ii) For any $v = y_1^k w \in Y^*$, $k \geq 1$, $w \in Y^* \setminus y_1 Y^*$, one has

$$\zeta_{\sqcup}(\pi_X \Sigma_v) + \sum_{m=2}^k B^{(m)} \zeta_{\sqcup}(\pi_X \Sigma_{y_1^{k-m} w}) = 0.$$

Proof. From Lemma 1, we see that $\langle B'(y_1) \mid y_1^0 \rangle = 1$, $\langle B'(y_1) \mid y_1 \rangle = 0$ and

$$\forall m \geq 2, \quad \langle B'(y_1) \mid y_1^m \rangle = B^{(m)}.$$

Using the basis $\{\Pi_w\}_{w \in Y^*}$ as a coordinate system, we identify the coefficients of the two sides in (42) and obtain the preceding statements. \square

Example 10. 1. For $v = y_2$, $\zeta(\Sigma_{y_2}) = \zeta(S_{x_0 x_1})$.

2. For $v = y_2 y_3$, $\zeta(\Sigma_{y_2 y_3}) = \zeta(S_{x_0 x_1 x_0^2 x_1}) - 2\zeta(S_{x_0^2 x_1 x_0 x_1}) - 2\zeta(S_{x_0^3 x_1^2}) + \zeta(S_{x_0^4 x_1})$.

3. For $v = y_1^3$, $-\frac{1}{2}\zeta(S_{x_0 x_1^2}) + \frac{1}{6}\zeta(S_{x_0^2 x_1}) + B^{(3)} = 0$.

4. For $v = y_1^2 y_2$, $\zeta(S_{x_0 x_1^3}) - \zeta(S_{x_0^2 x_1^2}) + \frac{1}{2}\zeta(S_{x_0^3 x_1}) + B^{(2)} = 0$.

3.2.2. Identifying with respect to the basis $\{P_w\}_{w \in X^*}$

Let us denote by¹⁸ $\{P'_w\}_{w \in X^* x_1}$ the reductions of $\{P_w\}_{w \in X^* x_1}$ on $\mathbb{Q} \oplus \mathbb{Q}\langle X \rangle_{x_1}$. By applying the mapping π_X on the two sides of (42) and using the duality of the bases, we can rewrite the regularization as follows

$$B'(x_1) \sum_{u \in X^* x_1} \zeta_{\sqcup}(S_u) P'_u = \sum_{u \in X^* x_1} \zeta_{\sqcup}(\pi_Y S_u) P'_u. \quad (43)$$

Similarly, remarking that $B'(x_1)$ is a series of a single letter, x_1 ,

$$x_1^k P_u = P_{x_1}^k P_u = P_{x_1^k u}, \quad \forall k \geq 1, u \in X^*.$$

Proposition 2. i) For any $u \in X^* \setminus x_1 X^*$, $\zeta(S_u) = \zeta(\pi_Y S_u)$.

ii) For any $u \in x_1 X^* \setminus x_1^2 X^*$, $\zeta_{\sqcup}(\pi_Y S_u) = 0$.

iii) For any $u = x_1^k w \in X^*$, $k \geq 2$, $w \in X^* \setminus x_1 X^*$, $B^{(k)} \zeta(S_w) = \zeta_{\sqcup}(\pi_Y S_u)$.

Proof. Similarly to Proposition 1, admitting the basis $\{P_w\}_{w \in X^*}$ as a coordinate system, we identify the coefficients of the two sides in (43) and then obtain the statements. \square

Example 11. 1. For $u = x_0 x_1$, $\zeta(S_{x_0 x_1}) = \zeta(\Sigma_{y_2})$.

2. For $u = x_0 x_1 x_0^2 x_1$, $\zeta(S_{x_0 x_1 x_0^2 x_1}) = \zeta(\Sigma_{y_2 y_3}) + 2\zeta(\Sigma_{y_3 x_2}) + 6\zeta(\Sigma_{y_4 x_1}) - 5\zeta(\Sigma_{y_5})$.

3. For $u = x_1 x_0 x_1$, $\zeta(\Sigma_{y_2 y_1}) - \frac{3}{2}\zeta(\Sigma_{y_3}) = 0$.

4. For $u = x_1^2 x_0 x_1$, $B^{(2)} \zeta(S_{x_0 x_1}) = 2\zeta(\Sigma_{y_4}) - \zeta(\Sigma_{y_2})^2 - \zeta(\Sigma_{y_3 y_1})$.

¹⁸They are defined by $P'_w = \pi_X(\pi_Y P_w)$, $\forall w \in X^*$. Note that $\pi_Y P_w = \pi_Y w = 0$, $\forall w \in X^* x_0$.

3.3. Algorithms to represent the structure of polyzetas

From Proposition 1 and 2, we really have relations among polyzetas represented on the bases $\{S_w\}_{w \in X^*}$ and $\{\Sigma_w\}_{w \in Y^*}$.

In fact, thanks to the formulas (30) and (38), we can easily represent these relations on the pure transcendence bases $\{S_l\}_{l \in \mathcal{L}ynX}$ or $\{\Sigma_l\}_{l \in \mathcal{L}ynY}$ respectively.

In the two following algorithms, one uses these relations and the other one (*Algorithm 3*) uses as well the structures of shuffle and stuffle products, we will eliminate these relations, in weight, to find the structure of polyzetas represented on the bases $\{S_l\}_{l \in \mathcal{L}ynX}$ and $\{\Sigma_l\}_{l \in \mathcal{L}ynY}$. The following two algorithms will be proceeded by recurrence on the weight of the words.

The same result obtained will be shown in the next subsection.

Algorithm 2

This algorithm uses Proposition 1 and *Algorithm 1* to establish polynomial relations among polyzetas on the basis $\{S_l\}_{l \in \mathcal{L}ynX}$ or uses Proposition 2 and *Algorithm 1* to establish relations among polyzetas on the basis $\{\Sigma_l\}_{l \in \mathcal{L}ynY}$.

We display here the second case.

INPUT: A positive integer n .

OUTPUT: The representations of polyzetas of weight n in terms of irreducible elements of polyzetas on the transcendence basis $\{\Sigma_l\}_{l \in \mathcal{L}ynY}$.

Step 1. We set the list, denoted by X_n , of all words¹⁹ of weight²⁰ n of X^*x_1 .

Step 2. For each $w \in X_n$, we set the polynomial $\mathcal{P} := \pi_Y(S_w)$ in $\mathbb{Q}\langle Y \rangle$ and thanks to *Algorithm 1* we represent $\zeta(\mathcal{P})$ in terms of $\{\zeta(\Sigma_l)\}_{l \in \mathcal{L}ynY}$. By taking the representations of $\zeta(\Sigma_l)$'s from the data of lower weights, we make representation in terms of irreducible elements for $\zeta(\mathcal{P})$ and proceed to establish a polynomial relation as follows:

- i) If $w \in \mathcal{L}ynX$ then we store $\zeta(\mathcal{P})$ to the variable $\zeta(S_w)$,
- ii) If $w = x_1u, u \in x_0X^*x_1$ then we make the relation $\zeta(\mathcal{P}) = 0$.
- iii) If $w \in x_0X^*x_1 \setminus \mathcal{L}ynX$, we rewrite w in the form of Lyndon factorization, $w = l_1^{i_1} \dots l_k^{i_k}$. By taking $\zeta(S_{l_j}), j = 1 \dots k$ from the data of lower weights, we make the relation

$$\frac{1}{i_1! \dots i_k!} \zeta(S_{l_1})^{i_1} \dots \zeta(S_{l_k})^{i_k} = \zeta(\mathcal{P}).$$

¹⁹Note that, there are 2^{n-1} words of weight n .

²⁰In the alphabet X , the weight of a word is understood as the length of that word.

Step 3. We reduce the above relations to representations of polyzetas in terms of irreducible elements.

The next lemma will give another way to find the relations among the family $\{\zeta_{\sqcup}(S_w)\}_{w \in X^*}$ and the family $\{\zeta_{\boxplus}(\Sigma_w)\}_{w \in Y^*}$.

Lemma 2. *i) For any $l_1, l_2 \in \mathcal{Lyn}X \setminus X$ (resp. $l_1, l_2 \in \mathcal{Lyn}Y \setminus \{y_1\}$), one has*

$$\begin{aligned}\zeta(S_{l_1} \sqcup S_{l_2}) &= \zeta(\pi_Y(S_{l_1}) \boxplus \pi_Y(S_{l_2})), \\ \zeta(\Sigma_{l_1} \boxplus \Sigma_{l_2}) &= \zeta(\pi_X(\Sigma_{l_1}) \sqcup \pi_X(\Sigma_{l_2})).\end{aligned}$$

*ii) For any $w \in x_0X^*x_1$ or $w \in x_1x_0X^*x_1$ (resp. $w \in Y^* \setminus y_1^2Y^*$), one has*

$$\begin{aligned}\zeta_{\sqcup}(S_w) &= \zeta_{\boxplus}(\pi_Y(S_w)), \\ \zeta_{\boxplus}(\Sigma_w) &= \zeta_{\sqcup}(\pi_X(\Sigma_w)).\end{aligned}$$

Proof. Remark that, for any $w \in X^*$, $S_w = w + \sum_{v < w} \alpha_v v$ and if $l \in \mathcal{Lyn}X \setminus X$ then $l \in x_0X^*x_1$.

Relying on properties of polyzetas on words, i.e. [? ?]

$$\begin{aligned}\zeta(l_1 \sqcup l_2) &= \zeta(\pi_Y(l_1) \boxplus \pi_Y(l_2)), \quad \forall l_1, l_2 \in \mathcal{Lyn}X \setminus X, \\ \zeta_{\sqcup}(x_1 \sqcup l) &= \zeta_{\boxplus}(y_1 \boxplus \pi_Y(l)), \quad \forall l \in \mathcal{Lyn}X \setminus X,\end{aligned}$$

we get the expected results. \square

Example 12. For $l_1 = x_0x_1, l_2 = x_0^2x_1^2$ (in $\mathcal{Lyn}X$) and $l_1 = y_2, l_2 = y_3y_1$ (in $\mathcal{Lyn}Y$):

$$\begin{aligned}\zeta(S_{x_0x_1})\zeta(S_{x_0^2x_1^2}) &= \zeta(\Sigma_{y_2})\zeta(\Sigma_{y_3y_1}) - \frac{1}{2}\zeta(\Sigma_{y_2})\zeta(\Sigma_{y_4}), \\ \zeta(\Sigma_{y_2})\zeta(\Sigma_{y_3y_1}) &= \zeta(S_{x_0x_1})\zeta(S_{x_0^2x_1^2}) + \frac{1}{2}\zeta(S_{x_0x_1})\zeta(S_{x_0^3x_1}).\end{aligned}$$

For $w = x_1x_0^2x_1$ (in $x_1x_0X^*x_1$) and $w = y_1y_3$ (in y_1Y^*):

$$\begin{aligned}0 &= \frac{1}{2}\zeta(\Sigma_{y_2})^2 + \zeta(\Sigma_{y_3y_1}) - 2\zeta(\Sigma_{y_4}), \\ 0 &= -\frac{1}{2}\zeta(S_{x_0x_1})^2 + \zeta(S_{x_0^2x_1^2}) + \zeta(S_{x_0^3x_1}).\end{aligned}$$

Algorithm 3

This algorithm uses Lemma 2 and *Algorithm 1* to establish polynomial relations among polyzetas on the basis $\{S_l\}_{l \in \mathcal{L}ynX}$ or the basis $\{\Sigma_l\}_{l \in \mathcal{L}ynY}$.

We display here the second case.

INPUT: A positive integer n .

OUTPUT: The representations of polyzetas of weight n in terms of irreducible elements of polyzetas on the transcendence basis $\{\Sigma_l\}_{l \in \mathcal{L}ynY}$.

Step 1. We set a list, denoted by X_n , all words of weight n in $x_0X^*x_1$ or $x_1x_0X^*x_1$.

Step 2. We establish polynomial relations of weight n as follows. For each $w \in X_n$, we make a polynomial \mathcal{P} in $\mathbb{Q}\langle Y \rangle$ by the way:

- i) If $w \in \mathcal{L}ynX$ then $\mathcal{P} := \pi_Y(S_{l_1}) \boxminus \pi_Y(S_{l_2}) - \pi_Y(S_{l_1} \sqcup S_{l_2})$, where (l_1, l_2) is the standard factorization of w .
- ii) If $w = x_1w_1$ then $\mathcal{P} := \pi_Y(S_{x_1}) \boxminus \pi_Y(S_{w_1}) - \pi_Y(S_{x_1} \sqcup S_{w_1})$.
- iii) If $w = l_1^{i_1} \dots l_k^{i_k}$, $l_1, \dots, l_k \in \mathcal{L}ynX$, $l_1 > \dots > l_k$ then
 $\mathcal{P} := \pi_Y(S_{l_1})^{\boxminus i_1} \boxminus \dots \boxminus \pi_Y(S_{l_k})^{\boxminus i_k} - \pi_Y(S_{l_1} \sqcup \dots \sqcup S_{l_k})$.

Thanks to *Algorithm 1*, we represent $\zeta(\Sigma_{\mathcal{P}})$ in terms of $\{\zeta(\Sigma_l)\}_{l \in \mathcal{L}ynY}$ (here, $\zeta(\Sigma_l)$ are taken from the data of lower weights). At last, we make the relation $\zeta(\Sigma_{\mathcal{P}}) = 0$.

Step 3. We reduce the above relations to representations of polyzetas in terms of irreducible elements.

These algorithms produce homogeneous polynomial relations among local coordinates $\{\zeta(\Sigma_l)\}_{l \in \mathcal{L}ynY}$ (resp. $\{\zeta(S_l)\}_{l \in \mathcal{L}ynX}$). Each identity is indexed by a Lyndon word and is not an identity of the tautological form

$$\zeta(\Sigma_l) = \zeta(\Sigma_l) \quad (\text{or } \zeta(S_l) = \zeta(S_l)). \quad (44)$$

Replacing "=" by " \longrightarrow " in these homogeneous polynomial relations, we obtain a noetherian rewriting system among $\{\zeta(\Sigma_l)\}_{l \in \mathcal{L}ynY}$ (resp. $\{\zeta(S_l)\}_{l \in \mathcal{L}ynX}$) in which irreducible terms are polyzetas involved in tautologies (44) and they are viewed as algebraic generators of the algebra of convergent polyzetas [? ?].

3.4. Results

3.4.1. Representation of polyzetas in terms of irreducible polyzetas

The following results were computed by our package in Maple [?] thanks to *Algorithm 2* (or *Algorithm 3*). We show here representations of polyzetas in terms of irreducible polyzetas of the bases indexed by Lyndon words on the two alphabets X and Y .

For each weight n , the list of Lyndon words $l \in \mathcal{L}ynY$ will be displayed in the second column, and their projection over X , i.e. $\pi_X(l) \in \mathcal{L}ynX$, will be displayed in the fourth column which are also, due to a lemma by D. Perrin, the list of Lyndon words in $\mathcal{L}ynY$ (see Table 1).

n	l	$\zeta(\Sigma_l)$	$\pi_X(l)$	$\zeta(S_{\pi_X(l)})$
3	y_2y_1	$\frac{3}{2}\zeta(\Sigma_{y_3})$	$x_0x_1^2$	$\zeta(S_{x_0^2x_1})$
4	y_4	$\frac{2}{5}\zeta(\Sigma_{y_2})^2$	$x_0^3x_1$	$\frac{2}{5}\zeta(S_{x_0x_1})^2$
	y_3y_1	$\frac{3}{10}\zeta(\Sigma_{y_2})^2$	$x_0^2x_1^2$	$\frac{1}{10}\zeta(S_{x_0x_1})^2$
	$y_2y_1^2$	$\frac{2}{3}\zeta(\Sigma_{y_2})^2$	$x_0x_1^3$	$\frac{2}{5}\zeta(S_{x_0x_1})^2$
5	y_4y_1	$-\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) + \frac{5}{2}\zeta(\Sigma_{y_5})$	$x_0^3x_1^2$	$-\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1}) + 2\zeta(S_{x_0^4x_1})$
	y_3y_2	$3\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) - 5\zeta(\Sigma_{y_5})$	$x_0^2x_1x_0x_1$	$-\frac{3}{2}\zeta(S_{x_0^4x_1}) + \zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1})$
	$y_3y_1^2$	$\frac{5}{12}\zeta(\Sigma_{y_5})$	$x_0^2x_1^3$	$-\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1}) + 2\zeta(S_{x_0^4x_1})$
	$y_2^2y_1$	$\frac{3}{2}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) - \frac{25}{12}\zeta(\Sigma_{y_5})$	$x_0x_1x_0x_1^2$	$\frac{1}{2}\zeta(S_{x_0^4x_1})$
	$y_2y_1^3$	$\frac{1}{4}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) + \frac{5}{4}\zeta(\Sigma_{y_5})$	$x_0x_1^4$	$\zeta(S_{x_0^4x_1})$
6	y_6	$\frac{8}{35}\zeta(\Sigma_{y_2})^3$	$x_0^5x_1$	$\frac{8}{35}\zeta(S_{x_0x_1})^3$
	y_5y_1	$\frac{2}{7}\zeta(\Sigma_{y_2})^3 - \frac{1}{2}\zeta(\Sigma_{y_3})^2$	$x_0^4x_1^2$	$\frac{6}{35}\zeta(S_{x_0x_1})^3 - \frac{1}{2}\zeta(S_{x_0^2x_1})^2$
	y_4y_2	$\zeta(\Sigma_{y_3})^2 - \frac{4}{21}\zeta(\Sigma_{y_2})^3$	$x_0^3x_1x_0x_1$	$\frac{4}{105}\zeta(S_{x_0x_1})^3$
	$y_4y_1^2$	$\frac{3}{10}\zeta(\Sigma_{y_2})^3 - \frac{3}{4}\zeta(\Sigma_{y_3})^2$	$x_0^3x_1^3$	$\frac{23}{70}\zeta(S_{x_0x_1})^3 - \zeta(S_{x_0^2x_1})^2$
	$y_3y_2y_1$	$3\zeta(\Sigma_{y_3})^2 - \frac{9}{10}\zeta(\Sigma_{y_2})^3$	$x_0^2x_1x_0x_1^2$	$\frac{2}{105}\zeta(S_{x_0x_1})^3$
	$y_3y_1y_2$	$-\frac{17}{30}\zeta(\Sigma_{y_2})^3 + \frac{9}{4}\zeta(\Sigma_{y_3})^2$	$x_0^2x_1^2x_0x_1$	$-\frac{89}{210}\zeta(S_{x_0x_1})^3 + \frac{3}{2}\zeta(S_{x_0^2x_1})^2$
	$y_3y_1^3$	$\frac{1}{21}\zeta(\Sigma_{y_2})^3$	$x_0^2x_1^4$	$\frac{6}{35}\zeta(S_{x_0x_1})^3 - \frac{1}{2}\zeta(S_{x_0^2x_1})^2$
	$y_2^2y_1^2$	$\frac{11}{63}\zeta(\Sigma_{y_2})^3 - \frac{1}{4}\zeta(\Sigma_{y_3})^2$	$x_0x_1x_0x_1^3$	$\frac{8}{21}\zeta(S_{x_0x_1})^3 - \zeta(S_{x_0^2x_1})^2$
	$y_2y_1^4$	$\frac{17}{50}\zeta(\Sigma_{y_2})^3 + \frac{3}{16}\zeta(\Sigma_{y_3})^2$	$x_0x_1^5$	$\frac{8}{35}\zeta(S_{x_0x_1})^3$

Table 1: Representation of polyzetas in terms of irreducible polyzetas up to weight 6.

3.4.2. Conclusion of the results

Let us denote by \mathcal{Z}_n the \mathbb{Q} -vector space generated by polyzetas of weight n and d_n its dimension.

From the above representations, we obtain their bases as follows:

- $n = 2, d_2 = 1, \mathcal{Z}_2 = \text{span}_{\mathbb{Q}}\{\zeta(\Sigma_{y_2})\} = \text{span}_{\mathbb{Q}}\{\zeta(S_{x_0x_1})\}$
- $n = 3, d_3 = 1, \mathcal{Z}_3 = \text{span}_{\mathbb{Q}}\{\zeta(\Sigma_{y_3})\} = \text{span}_{\mathbb{Q}}\{\zeta(S_{x_0^2x_1})\}$
- $n = 4, d_4 = 1, \mathcal{Z}_4 = \text{span}_{\mathbb{Q}}\{\zeta(\Sigma_{y_2})^2\} = \text{span}_{\mathbb{Q}}\{\zeta(S_{x_0x_1})^2\}$
- $n = 5, d_5 = 2,$
 $\mathcal{Z}_5 = \text{span}_{\mathbb{Q}}\{\zeta(\Sigma_{y_5}), \zeta(\Sigma_{y_2})\zeta(\Sigma_{y_3})\} = \text{span}_{\mathbb{Q}}\{\zeta(S_{x_0^4x_1}), \zeta(S_{x_0x_1})\zeta(S_{x_0^2x_1})\}$
- $n = 6, d_6 = 2,$
 $\mathcal{Z}_6 = \text{span}_{\mathbb{Q}}\{\zeta(\Sigma_{y_2})^3, \zeta(\Sigma_{y_3})^2\} = \text{span}_{\mathbb{Q}}\{\zeta(S_{x_0x_1})^3, \zeta(S_{x_0^2x_1})^2\}$
- $n = 7, d_7 = 3,$
 $\mathcal{Z}_7 = \text{span}_{\mathbb{Q}}\{\zeta(\Sigma_{y_7}), \zeta(\Sigma_{y_2})\zeta(\Sigma_{y_5}), \zeta(\Sigma_{y_2})^2\zeta(\Sigma_{y_3})\}$
 $= \text{span}_{\mathbb{Q}}\{\zeta(S_{x_0^6x_1}), \zeta(S_{x_0x_1})\zeta(S_{x_0^4x_1}), \zeta(S_{x_0x_1})^2\zeta(S_{x_0^2x_1})\}$
- $n = 8, d_8 = 4,$
 $\mathcal{Z}_8 = \text{span}_{\mathbb{Q}}\{\zeta(\Sigma_{y_2})^4, \zeta(\Sigma_{y_3y_1^5}), \zeta(\Sigma_{y_3})\zeta(\Sigma_{y_5}), \zeta(\Sigma_{y_3})^2\zeta(\Sigma_{y_2})\}$
 $= \text{span}_{\mathbb{Q}}\{\zeta(S_{x_0x_1})^4, \zeta(S_{x_0^2x_1})\zeta(S_{x_0^4x_1}), \zeta(S_{x_0x_1})\zeta(S_{x_0^2x_1})^2,$
 $\zeta(S_{x_0x_1^2x_0x_1^4})\}$
- $n = 9, d_9 = 5,$
 $\mathcal{Z}_9 = \text{span}_{\mathbb{Q}}\{\zeta(\Sigma_{y_9}), \zeta(\Sigma_{y_2})^2\zeta(\Sigma_{y_5}), \zeta(\Sigma_{y_2})\zeta(\Sigma_{y_7}), \zeta(\Sigma_{y_2})^3\zeta(\Sigma_{y_3}),$
 $\zeta(\Sigma_{y_3})^3\}$
 $= \text{span}_{\mathbb{Q}}\{\zeta(S_{x_0^8x_1}), \zeta(S_{x_0x_1})^2\zeta(S_{x_0^4x_1}), \zeta(S_{x_0^2x_1})^3, \zeta(S_{x_0x_1})\zeta(S_{x_0^6x_1}),$
 $\zeta(S_{x_0x_1})^3\zeta(S_{x_0^2x_1})\}$
- $n = 10, d_{10} = 7,$
 $\mathcal{Z}_{10} = \text{span}_{\mathbb{Q}}\{\zeta(\Sigma_{y_2})^5, \zeta(\Sigma_{y_5})^2, \zeta(\Sigma_{y_3y_1^7}), \zeta(\Sigma_{y_2})\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_5}),$
 $\zeta(\Sigma_{y_2})^2\zeta(\Sigma_{y_3})^2, \zeta(\Sigma_{y_3})\zeta(\Sigma_{y_7}), \zeta(\Sigma_{y_2})\zeta(\Sigma_{y_3y_1^5})\}$
 $= \text{span}_{\mathbb{Q}}\{\zeta(S_{x_0^4x_1})^2, \zeta(S_{x_0^4x_1})\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1}), \zeta(S_{x_0x_1})^2\zeta(S_{x_0^2x_1})^2,$
 $\zeta(S_{x_0x_1})^5, \zeta(S_{x_0x_1^3x_0x_1^5}), \zeta(S_{x_0^6x_1})\zeta(S_{x_0^2x_1}), \zeta(S_{x_0x_1})\zeta(S_{x_0x_1^2x_0x_1^4})\}$

- $n = 11, d_{11} = 9,$

$$\begin{aligned}
\mathcal{Z}_{11} &= \text{span}_{\mathbb{Q}}\{\zeta(\Sigma_{y_{11}}), \zeta(\Sigma_{y_2})^2\zeta(\Sigma_{y_7}), \zeta(\Sigma_{y_2})\zeta(\Sigma_{y_9}), \zeta(\Sigma_{y_2})^3\zeta(\Sigma_{y_5}), \\
&\quad \zeta(\Sigma_{y_2y_1^9}), \zeta(\Sigma_{y_3})^2\zeta(\Sigma_{y_5}), \zeta(\Sigma_{y_2})\zeta(\Sigma_{y_3})^3, \zeta(\Sigma_{y_2})^2\zeta(\Sigma_{y_7}), \\
&\quad \zeta(\Sigma_{y_3})\zeta(\Sigma_{y_3y_1^5})\} \\
&= \text{span}_{\mathbb{Q}}\{\zeta(S_{x_0^{10}x_1}), \zeta(S_{x_0^4x_1})\zeta(S_{x_0^2x_1})^2, \zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1})^3\zeta(S_{x_0x_1}), \\
&\quad \zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1})^4, \zeta(S_{x_0^4x_1})\zeta(S_{x_0x_1})^3, \zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1^2x_0x_1^4}), \\
&\quad \zeta(S_{x_0x_1^2x_0x_1^2x_0x_1^4}), \zeta(S_{x_0^6x_1})\zeta(S_{x_0x_1})^2, \zeta(S_{x_0^8x_1})\zeta(S_{x_0x_1})\}
\end{aligned}$$

- $n = 12, d_{12} = 12,$

$$\begin{aligned}
\mathcal{Z}_{12} &= \text{span}_{\mathbb{Q}}\{\zeta(\Sigma_{y_2})^6, \zeta(\Sigma_{y_3})^4, \zeta(\Sigma_{y_2})\zeta(\Sigma_{y_5})^2, \zeta(\Sigma_{y_3}\Sigma_{y_1^9}), \zeta(\Sigma_{y_2}^2\Sigma_{y_1^8}), \\
&\quad \zeta(\Sigma_{y_2})^3\zeta(\Sigma_{y_3})^2, \zeta(\Sigma_{y_3})\zeta(\Sigma_{y_9}), \zeta(\Sigma_{y_2})\zeta(\Sigma_{y_3}\Sigma_{y_1^7}), \zeta(\Sigma_{y_5})\zeta(\Sigma_{y_7}), \\
&\quad \zeta(\Sigma_{y_2})\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_7}), \zeta(\Sigma_{y_2})^2\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_5}), \zeta(\Sigma_{y_2})^2\zeta(\Sigma_{y_3}\Sigma_{y_1^5})\} \\
&= \text{span}_{\mathbb{Q}}\{\zeta(S_{x_0x_1})^6, \zeta(S_{x_0^2x_1})^4, \zeta(S_{x_0x_1x_0x_1^9}), \zeta(S_{x_0x_1})^2\zeta(S_{x_0x_1^2x_0x_1^4}), \\
&\quad \zeta(S_{x_0^4x_1})\zeta(S_{x_0^6x_1}), \zeta(S_{x_0x_1})^3\zeta(S_{x_0^2x_1})^2, \zeta(S_{x_0x_1})^2\zeta(S_{x_0^2x_1})\zeta(S_{x_0^4x_1}), \\
&\quad \zeta(S_{x_0x_1})\zeta(S_{x_0^2x_1})\zeta(S_{x_0^6x_1}), \zeta(S_{x_0^2x_1})\zeta(S_{x_0^8x_1}), \zeta(S_{x_0x_1})\zeta(S_{x_0^4x_1})^2, \\
&\quad \zeta(S_{x_0x_1})\zeta(S_{x_0x_1^3x_0x_1^5}), \zeta(S_{x_0^3x_1x_0x_1^7})\}.
\end{aligned}$$

We can see that these dimensions satisfy the following recurrence [?]]

$$d_1 = 0, d_2 = d_3 = 1 \quad \text{and} \quad \forall n \geq 4, d_n = d_{n-2} + d_{n-3}.$$

This means that, up to weight 12, our results obtained by the previous algorithms verify the Zagier's dimension conjecture. As a consequence, this conjecture holds up to weight 12 if and only if the irreducible polyzetas, contained in each two

following different lists, are algebraically independent (see [?] for a discussion).

n	irreducible polyzetas on $\{\Sigma_l\}_{l \in \mathcal{L}ynY}$	irreducible polyzetas on $\{S_l\}_{l \in \mathcal{L}ynX}$
2	$\zeta(\Sigma_{y_2})$	$\zeta(S_{x_0x_1})$
3	$\zeta(\Sigma_{y_3})$	$\zeta(S_{x_0^2x_1})$
4		
5	$\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^4x_1})$
6		
7	$\zeta(\Sigma_{y_7})$	$\zeta(S_{x_0^6x_1})$
8	$\zeta(\Sigma_{y_3y_1^5})$	$\zeta(S_{x_0x_1^2x_0x_1^4})$
9	$\zeta(\Sigma_{y_9})$	$\zeta(S_{x_0^8x_1})$
10	$\zeta(\Sigma_{y_3y_1^7})$	$\zeta(S_{x_0x_1^2x_0x_1^6})$
11	$\zeta(\Sigma_{y_{11}}), \zeta(\Sigma_{y_2y_1^9})$	$\zeta(S_{x_0^{10}x_1}), \zeta(S_{x_0x_1^2x_0x_1^2x_0x_1^4})$
12	$\zeta(\Sigma_{y_2^2y_1^8}), \zeta(\Sigma_{y_3y_1^9})$	$\zeta(S_{x_0x_1x_0x_1^9}), \zeta(S_{x_0^3x_1x_0x_1^7})$

Table 2: List of irreducible polyzetas up to weight 12.

By Example 6, in general, one has

$$\forall l \in \mathcal{L}ynY, \pi_X(\Sigma_l) \neq S_{\pi_X l} \quad \text{and} \quad \forall l \in \mathcal{L}ynX \setminus \{x_0\}, \pi_Y(S_l) \neq \Sigma_{\pi_Y l}.$$

This does not occur, due to a lemma by D. Perrin, with the Lyndon words themselves on which $\{\zeta(l)\}_{l \in \mathcal{L}ynY}$ (or $\{\zeta(l)\}_{l \in \mathcal{L}ynX}$) was provided in [? ? ?]. Hence, we insist on the fact that $\{\zeta(\Sigma_l)\}_{l \in \mathcal{L}ynY}$ and $\{\zeta(S_l)\}_{l \in \mathcal{L}ynX}$ provide two different systems of local coordinates and two lists of irreducible polyzetas (see Table 2).

4. Conclusion

In the classical theory of (finite-dimensional) Lie groups, every ordered basis of the Lie algebra provides a system of local coordinates of a suitable neighbourhood of the unity (of the group) via an ordered product of one-parameter groups corresponding to the (ordered) basis.

Here, we get a perfect analogue of this geometrical picture for the Hausdorff groups (in shuffle and stuffle Hopf algebras) through Schützenberger’s factorization. This does not depend on the regularization of shuffle and quasi-shuffle.

Moreover, through the bridge equation (6) which relates two elements on these groups and an identification of the local coordinates of the L.H.S. and R.H.S. of (7) which involve only convergent polyzetas as local coordinates, we get, up to weight 12,

- a confirmation of the Zagier's dimension conjecture,
- two families of irreducible polyzetas (*i.e* two algebraic bases for polyzetas),

which are not due to the regularized double-shuffle relations (and we do not need any regularization).

This implementation will be used, in our forthcoming work, to determine the asymptotic expansions of harmonic sums via Euler-Maclaurin formula.

Bibliography